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Functional Approach to Scattering in Quantum Mechanics

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A functional approach to scattering theory in quantum mechanics is developed by deriving an explicit functional expression for *transition* amplitudes. In applications, the formalism avoids dealing with noncommutativity problems of operators, solving the Schrödinger equation (or the integral equation of the Green's function), or dealing with the often quite complicated continual (path) integrals and, most importantly, applies to short- and long-range interactions. The basic idea is the use of the quantum action principle followed by a systematic analysis of the concept of an intervening source developed earlier in the study of stimulated emission. A comparison with the standard approach is also made.

1. INTRODUCTION

The purpose of this paper is to derive an explicit functional expression for *transition* amplitudes for scattering in quantum mechanics. The method avoids dealing with noncommutativity properties of operators usually encountered. It avoids solving the Schrödinger equation or its related integral for the Green's function for each particular potential. It avoids dealing with the often complicated continual (Feynman and Hibbs, 1965; Duru and Kleinert, 1982; Manoukian, 1985) path integrals. And most importantly, it applies to short- and long-range (Dollard, 1964; Weinberg, 1965; Schweber, 1973; Soffer, 1983; Manoukian, 1984, 1985, 1986a) interactions. [By a long-range interaction one means a dynamics involving a potential that vanishes like $O(1/r)$ or slower for $|\mathbf{r}|\to\infty$.] For example, it avoids the difficulty encountered with the so-called infinite Dalitz phase factor in the Coulomb problem (Weinberg, 1965; Manoukian, 1986a). Our method uses the quantum action principle (Schwinger, 1951; see also Schwinger, 1960; Lam, 1965; Manoukian, 1985) or its modification

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(Manoukian, 1985) followed by a systematic analysis of the concept of an intervening source arising in the study of stimulated emission carried out in an earlier paper (Manoukian, 1986b). In Section 2 the functional formalism is developed for short-range potentials and in Section 3 a comparison with the standard approach is made. Section 4 generalizes the method of Section 2 to a long-range interaction with an explicit application to the Coulomb problem. In the concluding section (Section 5) we spell out advantages of our formalism over the orthodox one. The extension of this work to quantum field theory will be attempted elsewhere.

2. SHORT-RANGE INTERACTIONS

The Hamiltonian may be conveniently written in a second-quantized formalism $[x^0 = t, x = (x^0, \mathbf{x})]$:

$$H(t) = \int d^3\mathbf{x} \left\{ \phi^\dagger(x) \left[-\frac{\nabla^2}{2m} + \lambda V(\mathbf{x}) \right] \phi(x) - K^*(x)\phi(x) - \phi^\dagger(x) K(x) \right\} \tag{1}$$

where $\lambda V(\mathbf{x})$ is the potential, $K(x)$ is an external c-number source of compact support in time, and $\delta K(x)/\delta K(x') = \delta^4(x - x')$. Scattering in and out states will be denoted (Manoukian, 1985), respectively, by $|gT'; 0, \lambda, K\rangle$ and $|fT; 0, \lambda, K\rangle$, where we eventually take $T' \to -\infty$ and $T \to \infty$, to obtain for the latter $|g_-\rangle = |g - \infty; 0, \lambda, K\rangle$ and $|f_+\rangle = |f + \infty; 0, \lambda, K\rangle$. The quantum action principle (Schwinger, 1951, 1960; Lam, 1965; Manoukian, 1985) then reads

$$\frac{\partial}{\partial \lambda} \langle f_+ | g_- \rangle^{K,\lambda} = i \int (dx) \frac{\delta}{\delta K(x)} V(\mathbf{x}) \frac{\delta}{\delta K^*(x)} \langle f_+ | g_- \rangle^{K,\lambda} \tag{2}$$

where $(dx) = dx^0\ dx^1\ dx^2\ dx^3$. Equation (2) is then integrated out with respect to λ, to lead, for $K = 0$, $K^* = 0$, to

$$\langle f_+ | g_- \rangle = \exp\left[i\lambda \int (dx) \frac{\delta}{\delta K(x)} V(\mathbf{x}) \frac{\delta}{\delta K^*(x)} \right] \langle f_+ | g_- \rangle^{K,0} \big|_{K=0,\, K^*=0} \tag{3}$$

where $\langle f_+ | g_- \rangle = \langle f_+ | g_- \rangle^{0,\lambda}$, and $\langle f_+ | g_- \rangle^{K,0}$ denotes $\langle f_+ | g_- \rangle^{K,\lambda}$ with λ set equal to zero and depends explicitly on the source K. In particular, for a particle with incoming and outgoing momenta $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively, we have

$$\langle \mathbf{p}_2 + | \mathbf{p}_1 - \rangle = \exp\left[i\lambda \int (dx) \frac{\delta}{\delta K(x)} V(\mathbf{x}) \frac{\delta}{\delta K^*(x)} \right] \langle \mathbf{p}_2 + | \mathbf{p}_1 - \rangle^{K,0} \big|_{K=0,\, K^*=0} \tag{4}$$

Hence the basic problem is to find the expression for $\langle \mathbf{p}_2 + | \mathbf{p}_1 - \rangle^{K,0}$, which depends explicitly on K. This will be obtained directly from our earlier analysis (Manoukian, 1986b) of stimulated emission by an intervening source. To this end we write the external source as (Schwinger, 1970; Manoukian, 1986b) $K(x)=K_1(x)+K_2(x)+K_3(x)$, where K_1 is an emission source and K_3 is a detection source switched on after the intervening source K_2 is switched off, and K_2 is switched on after K_1 is switched off. The vacuum-to-vacuum transition amplitude for $\lambda=0$, in the nonrelativistic context, may be then written as

$$\langle 0_+|0_-\rangle^K=\left(\prod_{j=1}^{3}\langle 0_+|0_-\rangle^{K_j}\right)\exp(iK_3^*iK_2)\exp(iK_3^*iK_1)\exp(iK_2^*iK_1) \tag{5}$$

where

$$\begin{aligned}\langle 0_+|0_-\rangle^{K_j}&=\exp\left[i\int(dx)\,(dx')\,K_j^*(x)G_+^0(x-x')K_j(x')\right]\\ &\equiv\exp(iK_j^*G_+^0K_j),\qquad j=1,2,3\end{aligned} \tag{6}$$

$$G_+^0(x-x')=\begin{cases} i\displaystyle\int\frac{d^2\mathbf{k}}{(2\pi)^3}\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]\exp\frac{-i\mathbf{k}^2(x^0-x'^0)}{2m}, & x^0>x'^0\\ 0, & x^0<x'^0\end{cases} \tag{7}$$

$$iK_3^*iK_2=\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d^3\mathbf{k}'}{(2\pi)^3}\,iK_3^*(\mathbf{k})[(2\pi)^3\delta(\mathbf{k}-\mathbf{k}')]iK_2(\mathbf{k}') \tag{8}$$

$$K_j(\mathbf{k})=K_j(k^0,\mathbf{k})|_{k^0=\mathbf{k}^2/2m}=K_j(k)|_{k^0=\mathbf{k}^2/2m} \tag{9}$$

$$K_j(k)=\int(dx)\,e^{-ikx}K_j(x) \tag{10}$$

The amplitudes that the emission source K_1 emits a particle with momentum $\mathbf{k}_1$ and the detection source K_3 detects it with a momentum $\mathbf{k}_2$ are given, respectively, by (Schwinger, 1970; Manoukian, 1986b)

$$\langle\mathbf{p}_1|0_-\rangle^{K_1}=\langle 0_+|0_-\rangle^{K_1}\left[\frac{d^3\mathbf{p}_1}{(2\pi)^3}\right]^{1/2}iK_1(\mathbf{p}_1) \tag{11}$$

$$\langle 0_+|\mathbf{p}_2\rangle^{K_3}=\langle 0_+|0_-\rangle^{K_3}\left[\frac{d^3\mathbf{p}_2}{(2\pi)^3}\right]^{1/2}iK_3^*(\mathbf{p}_2) \tag{12}$$

A unitarity expansion of $\langle 0_+|0_-\rangle^K$ in (5) with respect to the sources K_1, K_2, K_3, as arranged causally, then leads (Manoukian, 1986b) from (5) and (8) for

$$\langle 0_+|\mathbf{p}_2+\rangle^{K_3}\langle\mathbf{p}_2+|\mathbf{p}_1-\rangle^{K_2}\langle\mathbf{p}_1-|0_-\rangle^{K_1}$$

to the expression

$$\left(\prod_{j=1}^{3}\langle 0_+|0_-\rangle^{K_j}\right)\frac{d^3\mathbf{p}_1}{(2\pi)^3}\frac{d^3\mathbf{p}_2}{(2\pi)^3}\,iK_3^*(\mathbf{p}_2)[(2\pi)^3\delta(\mathbf{p}_2-\mathbf{p}_1) + iK_2(\mathbf{p}_2)iK_2^*(\mathbf{p}_1)]iK_1(\mathbf{p}_1) \tag{13}$$

From (11) and (12) we then obtain ($\lambda=0$)

$$\langle \mathbf{p}_2+|\mathbf{p}_1-\rangle^{K_2}=\left[\frac{d^3\mathbf{p}_2}{(2\pi)^3}\right]^{1/2}\left[\frac{d^3\mathbf{p}_1}{(2\pi)^3}\right]^{1/2} \times[(2\pi)^3\delta(\mathbf{p}_1-\mathbf{p}_2)+iK_2(\mathbf{p}_2)iK_2^*(\mathbf{p}_1)]\langle 0_+|0_-\rangle^K \tag{14}$$

Using this expression in (4), we obtain for the transition amplitude

$$(\mathbf{p}_2+|\mathbf{p}_1-)=\exp\left[i\lambda\int(dx)\,\frac{\delta}{\delta K(x)}\,V(x)\,\frac{\delta}{\delta K^*(x)}\right] \times[(2\pi)^3\delta^3(\mathbf{p}_1-\mathbf{p}_2) + \int(dx)\,(dx')\,e^{-ip_2x}\,e^{ip_1x'}\,iK(x)iK^*(x')]\langle 0_+|0_-\rangle^K|_{K=0,\,K^*=0} \tag{15}$$

with $px=\mathbf{p}\cdot\mathbf{x}-p^0x^0$, $p_i^0=\mathbf{p}_i^2/2m$, $i=1,2$, and

$$\langle \mathbf{p}_2+|\mathbf{p}_1-\rangle=\left[\frac{d^3\mathbf{p}_1}{(2\pi)^3}\right]^{1/2}\left[\frac{d^3\mathbf{p}_2}{(2\pi)^3}\right]^{1/2}(\mathbf{p}_2+|\mathbf{p}_1-) \tag{16}$$

Equation (15) is an explicit expression for transition amplitudes for short-range interaction, obtained by taking functional derivatives, and will be generalized for long-range interactions in Section 4. We note that for $\lambda=0$ we satisfy the consistency relation $(\mathbf{p}_2+|\mathbf{p}_1-)=(2\pi)^3\delta^3(\mathbf{p}_1-\mathbf{p}_2)$. Other expressions for $(\mathbf{p}_2+|\mathbf{p}_1-)$, as obtained from (15), are given in the next section.

3. COMPARISON WITH THE STANDARD APPROACH

By using the translational operation

$$\exp\int(dx)\,f(x)\,\delta/\delta K(x)\,F[K]=F[K+f]$$

for an arbitrary functional $F[K]$ of K, we obtain, in reference to equation (15),

$$\exp\left(i\lambda\frac{\delta}{\delta K}\,V\frac{\delta}{\delta K^*}\right)K(x)K^*(x')\exp iK^*G_+^0K|_{K=0,\,K^*=0}$$

$$= \left[K^*(x') + i\lambda V(\mathbf{x}') \frac{\delta}{\delta K(x')} \right] \left[K(x) + i\lambda V(\mathbf{x}) \frac{\delta}{\delta K^*(x)} \right]$$

$$\times \exp i\lambda V \frac{\delta}{\delta K} \frac{\delta}{\delta K^*} \exp iK^* G_+^0 K|_{K=0,\, K^*=0}$$

$$= C\left[i\lambda V(\mathbf{x}')\, \delta^4(x-x') - \lambda^2 V(\mathbf{x}) V(\mathbf{x}') \frac{\delta}{\delta K(x')} \frac{\delta}{\delta K^*(x)} \right]$$

$$\times \exp iK^* G_+ K|_{K=0,\, K^*=0} \tag{17}$$

where C is a constant, independent of K and K^*, giving rise to the so-called closed loops, and G_+ denotes the exact Green's function satisfying

$$\left[-i\frac{\partial}{\partial t} - \frac{\nabla^2}{2m} + \lambda V(\mathbf{x}) \right] G_+(x-x') = \delta^4(x-x')$$
$$\left[i\frac{\partial}{\partial t'} - \frac{\nabla'^2}{2m} + \lambda V(\mathbf{x}') \right] G_+(x-x') = \delta^4(x-x') \tag{18}$$

We note, in particular, that the first term $i\lambda V(\mathbf{x}')\delta^4(x-x')$ in the square brackets on the right-hand side of (17) gives the classic *Born term* to the transition amplitude. Upon using (18), we obtain for the right-hand side of (17)

$$iC\left[\left(-i\frac{\partial}{\partial t} - \frac{\nabla^2}{2m} \right) \delta^4(x-x') - \left(-i\frac{\partial}{\partial t} - \frac{\nabla^2}{2m} \right)\left(i\frac{\partial}{\partial t'} - \frac{\nabla'^2}{2m} \right) G(x,x') \right] \tag{19}$$

When we replace (19) in (15), the first term in the square brackets in (19) gives zero contribution, thus leading to the *connected* amplitude, not involving the irrelevant closed loops, and the familiar expression

$$(\mathbf{p}_2 + |\mathbf{p}_1 -)_c = \left[(2\pi)^3 \delta^3(\mathbf{p}_1 - \mathbf{p}_2) + i \int (dx)\,(dx')\, e^{-ip_2 x}\, e^{ip_1 x'} \right.$$
$$\left. \times \left(-i\frac{\partial}{\partial t} - \frac{\nabla^2}{2m} \right)\left(i\frac{\partial}{\partial t'} - \frac{\nabla'^2}{2m} \right) G(x,x') \right] \tag{20}$$

involving the exact Green's function.

Another useful representation for $(\mathbf{p}_2 + |\mathbf{p}_1 -)$ as obtained from (20) is obtained by writing

$$G(x,x') = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{d^3\mathbf{k}}{(2\pi)^3} G(\mathbf{k}, x^0; \mathbf{k}', x'^0) \exp(i\mathbf{k}\cdot\mathbf{x}) \exp(-i\mathbf{k}'\cdot\mathbf{x}') \tag{21}$$

and using explicitly the boundary condition $G(\cdot, x^0, \cdot, x'^0) = 0$ for $x^0 < x'^0$ and hence, in particular, the conditions

$$\lim_{T'\to-\infty} G(\cdot, x^0, \cdot, T') = 0, \qquad \lim_{T\to\infty} G(\cdot, -T; \cdot, x'^0) = 0,$$

and the mass shell conditions $p_i^0=p_i^2/2m$, $i=1,2$, leading to the familiar expression

$$(\mathbf{p}_2+|\mathbf{p}_1-)_c=[(2\pi)^3\,\delta^3(\mathbf{p}_1-\mathbf{p}_2)-i\lim_{T\to\infty}\lim_{T'\to-\infty}\exp(i\mathbf{p}_2^2T/2m)$$
$$\times G(\mathbf{p}_2,T;\mathbf{p}_1T')\exp(-i\mathbf{p}_1^2T'/2m) \quad (22)$$

3. LONG-RANGE INTERACTIONS

The scattering states $|gT';0,\lambda,K\rangle$ and $|fT;0,\lambda,K\rangle$ develop in time via a unitary operator $U(t,\lambda,K)$

$$U(t,\lambda,K)|gT';0,\lambda,K\rangle=|gT';t,\lambda,K\rangle \quad (23)$$

$$U(t,\lambda,K)|fT;0,\lambda,K\rangle=|fT;t,\lambda,K\rangle \quad (24)$$

Asymptotically, for $t\to T'$, $T'\to-\infty$,

$$|gT';t,\lambda,K\rangle\to U_0(T',\lambda)|g\rangle \quad (25)$$

where, for long-range interactions, the unitary operator $U_0(T',\lambda)$ still depends on the quantum mechanical coupling λ (Dollard, 1964; Soffer, 1983; Manoukian, 1985) and is not simply given by $\exp(-iT'H_0)$, where H_0 is the free Hamiltonian. It is, however, independent of $K(x)$ due to the compact support nature of $K(x)$, for which we choose $K(x)$ to vanish for $x^0<T'$ and $x^0>T$. If we denote

$$U_0(t,\lambda)U_0^\dagger(t,\lambda+\delta\lambda)=1+i\delta\lambda G(t,\lambda,\phi^\dagger,\phi) \quad (26)$$

then with

$$|fT;0,\lambda\;K\rangle\equiv|fT;\lambda,K\rangle,\qquad |gT';0,\lambda,K\rangle\equiv|gT';\lambda,K\rangle$$

the modified quantum action principle reads (Manoukian, 1985)

$$\frac{\partial}{\partial\lambda}\langle fT;\lambda,K|gT';\lambda,K\rangle$$
$$=i\left[G\left(T,\lambda,-i\frac{\delta}{\delta K},-i\frac{\delta}{\delta K^*}\right)-G\left(T',\lambda,-i\frac{\delta}{\delta K},-i\frac{\delta}{\delta K^*}\right)\right]$$
$$\times i\int_{T'<x^0<T}(dx)\frac{\delta}{\delta K(x)}V(\mathbf{x})\frac{\delta}{\delta K^*(x)}\langle fT;\lambda,K|gT';\lambda,K\rangle$$

which, upon integration over λ, gives

$$\langle fT;\lambda,K|gT',\lambda,K\rangle$$
$$=\exp i\int_0^\lambda d\lambda'\left[G\left(T,\lambda',-i\frac{\delta}{\delta K},-i\frac{\delta}{\delta K^*}\right)-G\left(T',\lambda',-i\frac{\delta}{\delta K},-i\frac{\delta}{\delta K^*}\right)\right]$$
$$\times\exp i\lambda\int_{T'<x^0<T}(dx)\frac{\delta}{\delta K(x)}V(\mathbf{x})\frac{\delta}{\delta K^*(x)}\langle fT;0,K|gT';0,K\rangle \quad (27)$$

In particular, for a particle with ingoing and outgoing momenta $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively, we have

$$\begin{aligned}\langle \mathbf{p}_2 T; \lambda, K|\mathbf{p}_1 T'; \lambda, K\rangle &= \exp i\lambda \int_{T'<x^0<T} (dx)\frac{\delta}{\delta K(x)} V(\mathbf{x})\frac{\delta}{\delta K^*(x)} \\ &\times \exp i \int_0^\lambda d\lambda' \left[G\left(T, \lambda', -i\frac{\delta}{\delta K}, -i\frac{\delta}{\delta K^*}\right)\right. \\ &\left. - G\left(T', \lambda', -i\frac{\delta}{\delta K}, -i\frac{\delta}{\delta K^*}\right)\right]\langle \mathbf{p}_2 T; 0, K|\mathbf{p}_1 T'; 0, K\rangle \end{aligned} \tag{28}$$

The method of obtaining $G(t, \lambda, \cdot)$ is well known (Dollard, 1964; Dollard and Velo, 1966; Papanicolaou, 1974; Manoukian, 1985, 1986a) and very simple. We consider the expression

$$\langle \mathbf{p}_2 T; 0, K| \int_0^\lambda d\lambda'\, G\left(T', \lambda', -i\frac{\delta}{\delta K}, -i\frac{\delta}{\delta K^*}\right)|\mathbf{p}_1 T'; 0, K\rangle \tag{29}$$

Asymptotically for $T' \to -\infty$ the particle is essentially free with momentum $\mathbf{p}_1$ and hence we have $|\mathbf{x}/T'| \sim |\mathbf{p}_1|/m$. Therefore, if we formally replace $\mathbf{x}$ in $V(\mathbf{x})$ for $|\mathbf{x}| \to \infty$ by $T'\mathbf{p}_1/m$

$$\int^{T'} dt\, V(t\mathbf{p}_1/m) \to \int_0^\lambda \mathrm{d}\lambda'\, G(T', \lambda; -i\,\delta/\delta K; -i\,\delta/\delta K^*) \tag{30}$$

leading to the amplitude

$$\begin{aligned}\langle \mathbf{p}_2 T; \lambda, K|\mathbf{p}_1 T'; \lambda, K\rangle &= \exp i\lambda \int_{T'<x^0<T} (dx)\frac{\delta}{\delta K(x)} V \frac{\delta}{\delta K^*(x)} \\ &\times \exp i\left[\int^T dt\, V\left(\frac{t\mathbf{p}_2}{m}\right) - \int^{T'} dt\, V\left(\frac{t\mathbf{p}_1}{m}\right)\right]\langle \mathbf{p}_2 T; 0, K|\mathbf{p}_1 T'; 0, K\rangle \end{aligned} \tag{31}$$

Or, in its final form, we have

$$\begin{aligned}(\mathbf{p}_2 + |\mathbf{p}_1 -) &= \lim_{T' \to -\infty} \lim_{T \to \infty} \exp i\left[\int^T dt\, V\left(\frac{t\mathbf{p}_2}{m}\right) - \int^{T'} dt\, V\left(\frac{t\mathbf{p}_1}{m}\right)\right] \\ &\times \exp i\lambda \int dx \frac{\delta}{\delta K(x)} V(\mathbf{x})\frac{\delta}{\delta K^*(x)} \\ &\times \left[(2\pi)^3\, \delta(\mathbf{p}_1 - \mathbf{p}_2) + \int (dx) \int_{\substack{T'<x'^0<T \\ T'<x^0<T}} (dx')\, e^{-ip_2 x}\, e^{ip_1 x}\right. \\ &\left.\times iK(x) iK^*(x')\right]\langle 0_+|0_-\rangle^K|_{K=0,\,K^*=0} \end{aligned} \tag{32}$$

Equation (32) is our final expression for the transition amplitude for long- (and short-) range interactions. We first check that (32) coincides with (15) for short-range interactions defined by $V(\mathbf{x}) = O(|\mathbf{x}|^{-1-\delta})$ with $\delta > 0$. This gives

$$V(t\mathbf{p}/m) = O(|t|^{-1-\delta}), \qquad \int^{T} dt\, V(t\mathbf{p}/m) = O(|t|^{-\delta}) \to 0$$

and hence the correction terms (surface terms) in (32) do not contribute.

Now we apply (32) to Coulomb scattering. Here $V(\mathbf{x}) = e_1 e_2/|\mathbf{x}|$. Accordingly,

$$\int^{T} dt\, V(t\mathbf{p}/m) = (\operatorname{sgn} T)(e_1 e_2 m/|\mathbf{p}|) \ln|T| \tag{33}$$

giving

$$\begin{aligned}(\mathbf{p}_2 + |\mathbf{p}_1 -) = \lim_{T' \to -\infty} \lim_{T \to \infty} \Bigg\{ &\exp\left[i e_1 e_2 m \left(\frac{\ln|T|}{|\mathbf{p}_2|} + \frac{\ln|T'|}{|\mathbf{p}_1|} \right) \right] \\ &\times \exp i\lambda \int_{T' < x^0 < T} (dx) \frac{\delta}{\delta K(x)} V(\mathbf{x}) \frac{\delta}{\delta K^*(x)} \Bigg[(2\pi)^3\, \delta(\mathbf{p}_1 - \mathbf{p}_2) \\ &+ \int_{\substack{T' < x'^0 < T \\ T' < x^0 < T}} (dx) \int e^{-ip_2 x}\, e^{ip_1 x'} iK(x) iK^*(x') \Bigg] \\ &\times \langle 0_+ | 0_- \rangle^K |_{K=0,\, K^*=0}\end{aligned} \tag{34}$$

By an analysis similar to the one leading to (22), this may be rewritten as

$$\begin{aligned}(\mathbf{p}_2 + |\mathbf{p}_1 -)_c = \lim_{T' \to -\infty} \lim_{T \to \infty} &\exp i e_1 e_2 m \left(\frac{\ln|T|}{|\mathbf{p}_2|} + \frac{\ln|T'|}{|\mathbf{p}_1|} \right) \\ &\times \left[(2\pi)^3 \delta(\mathbf{p}_1 - \mathbf{p}_2) - i \exp \frac{i\mathbf{p}_2^2 T}{2m} G(\mathbf{p}_2, T; \mathbf{p}_1, T') \exp \frac{-i\mathbf{p}_1^2 T'}{2m} \right]\end{aligned} \tag{35}$$

When smeared with a test function in $\mathbf{p}_1$ and $\mathbf{p}_2$, the first term on the right-hand side of (35) will vanish by the Riemann-Lebesgue lemma for $|T|, |T'| \to \infty$. That is, in the sense of generalized functions, we may write

$$\begin{aligned}(\mathbf{p}_2 + |\mathbf{p}_1 -)_c = -i \lim_{T' \to -\infty} \lim_{T \to \infty} \Bigg[&\exp \frac{i\mathbf{p}_2^2 T}{2m} \exp \frac{i e_1 e_2 m \ln|T|}{|\mathbf{p}_2|} \\ &\times G(\mathbf{p}_2, T; \mathbf{p}_1, T') \exp \frac{i\mathbf{p}_2^2 T'}{2m} \exp \frac{i e_1 e_1 m \ln|T'|}{|\mathbf{p}_1|} \Bigg]\end{aligned} \tag{36}$$

The double limit $T' \to -\infty$, $T \to \infty$ in (36) exists and has been studied in detail by Papanicolaou (1974), obtained by completely different methods, and leads to an infrared finite expression (see also Manoukian, 1986a) without the so-called infinite Dalitz phase factor.

5. CONCLUSIONS

The functional expressions for transition amplitudes in (15) and (32) are quite general. Equation (32) is applicable to long-range interactions, thus avoiding infrared singularity problems. They avoid dealing with non-commutativity properties of operators and do not even involve the so-called creation and annihilation operators, as they give directly the expressions for the amplitudes with no additional work involved with operators. They avoid complications encountered with continual integrals, as they already give the solution to the latter in terms of functional differentiations. They do not involve explicitly the full Green's function and hence are suitable for perturbative treatments without having to go back to the integral equation of the Green's function. This approach will be generalized to quantum field theory and especially to gauge theories in a future report.

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